

## CAPILLARY SURFACES ARISING IN SINGULAR PERTURBATION PROBLEMS

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ABSTRACT. In this paper we prove Bernstein type theorems for a class of stationary points of the Alt-Caffarelli functional in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ . It is known that the stationary stable solutions (e.g. global minimizers) are smooth in  $\mathbb{R}^N$  for  $N \leq 4$ . However, this may not be the case for the stationary points obtained as limits of the singular perturbation problem in the unit ball  $B_1$

$$(0.1) \quad \begin{cases} \Delta u_\varepsilon(x) = \beta_\varepsilon(u_\varepsilon) & \text{in } B_1, \\ |u_\varepsilon| \leq 1 & \text{in } B_1, \end{cases}$$

where  $\beta_\varepsilon(t) = \frac{1}{\varepsilon}\beta(t/\varepsilon)$ ,  $\beta \in C_0^\infty[0, 1]$ ,  $\int_0^1 \beta(t)dt = M > 0$ , is an approximation of the Dirac measure and  $\varepsilon > 0$  is a parameter. The limit functions  $u = \lim_{\varepsilon \rightarrow 0} u_\varepsilon$  solve a Bernoulli type free boundary problem in a suitable weak sense. Our approach has two novelties: First we partition the free boundary  $\Gamma$  of a blow-up into three disjoint subsets which bear various properties of  $\Gamma$  and show that their components propagate in  $\Gamma$  instantaneously. An important tool we use is a new monotonicity formula for the solutions  $u_\varepsilon$  based on a computation of J. Spruck. It implies that any blow-up  $u_0$  of  $u$  either vanishes identically or is homogeneous function of degree one, that is  $u_0 = rg(\sigma)$ ,  $\sigma \in \mathbb{S}^{N-1}$  in the spherical coordinates. In particular, this implies that in two dimensions the singular set is empty at the non-degenerate points and in three dimensions the singular set of  $u_0$  is at most a singleton. As the Alt-Caffarelli example indicates this result cannot be improved. Second, we show that the spherical part  $g$  is the support function (in Minkowski's sense) of some capillary surface contained in the sphere of radius  $\sqrt{2M}$ . In particular, we show that  $\nabla u_0 : \mathbb{S}^2 \rightarrow \mathbb{R}^3$  is a conformal and minimal immersion and the Alt-Caffarelli example corresponds to a piece of catenoid which is a unique singular solution in  $\mathbb{R}^3$  determined by the support function  $g$  as a ring-type stationary minimal surface.

## 1. INTRODUCTION

In this paper we study the singular perturbation problem

$$(\mathcal{P}_\varepsilon) \quad \begin{cases} \Delta u_\varepsilon(x) = \beta_\varepsilon(u_\varepsilon) & \text{in } B_1, \\ |u_\varepsilon| \leq 1 & \text{in } B_1, \end{cases}$$

where  $\varepsilon > 0$  is small and

$$(1.1) \quad \begin{cases} \beta_\varepsilon(t) = \frac{1}{\varepsilon}\beta\left(\frac{t}{\varepsilon}\right), \\ \beta(t) \geq 0, \quad \text{supp } \beta \subset [0, 1], \\ \int_0^1 \beta(t)dt = M > 0, \end{cases}$$

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is an approximation of the Dirac measure,  $B_1 \subset \mathbb{R}^N$  is the unit ball centered at the origin. It is well known that  $(\mathcal{P}_\varepsilon)$  models propagation of equidiffusional premixed flames with high activation of energy [6]. Heuristically, the limit  $u_0 = \lim_{\varepsilon_j \rightarrow 0} u_{\varepsilon_j}$  (for a suitable sequence  $\varepsilon_j \rightarrow 0$ ) solves a Bernoulli type free boundary problem with the following free boundary condition

$$|\nabla u^+|^2 - |\nabla u^-|^2 = 2M.$$

If we choose  $\{u_\varepsilon\}$  to be a family of minimizers of

$$(1.2) \quad J_\varepsilon[u_\varepsilon] = \int_\Omega \frac{|\nabla u_\varepsilon|^2}{2} + \mathcal{B}(u_\varepsilon/\varepsilon), \quad \mathcal{B}(t) = \int_0^t \beta(s) ds,$$

then  $u_\varepsilon$  inherits the generic features of minimizers (e.g. non-degeneracy, rectifiability of  $\partial\{u > 0\}$ , etc.). Consequently, sending  $\varepsilon \rightarrow 0$  one can see that the limit  $u$  exists and it is a minimizer of the Alt-Caffarelli functional  $J[u] = \int_{B_1} |\nabla u|^2 + 2M\chi_{\{u>0\}}$ . It is known that the singular set of minimizers is empty in dimensions 2, 3 and 4, see [2], [7], [15]. However, if  $u_\varepsilon$  is not a minimizer then not much is known about the classification of the blow-ups of the limit function  $u$ .

This paper is devoted to the study of the blow-ups of the stationary points of the Alt-Caffarelli problem (obtained as limits of the singular perturbation problem  $(\mathcal{P}_\varepsilon)$ ) and establishes a new and direct connection with the capillary surfaces. In particular, we show that any blow-up of a limit function  $u = \lim_{\varepsilon_j \rightarrow 0} u_{\varepsilon_j}$  in  $\mathbb{R}^3$  (for an appropriate sequence  $\varepsilon_j$ ) defines a conformal and minimal immersion which is perpendicular to the sphere of radius  $\sqrt{2M}$  where  $M = \int_0^1 \beta(t) dt$ . In other words, one obtains a capillary surface in the sphere of radius  $\sqrt{2M}$ .

The first result in this paper is

**Theorem A.** *Let  $u_{\varepsilon_j} \rightarrow u$  locally uniformly in  $B_1$  for some subsequence  $\varepsilon_j$ , then any blow-up of  $u$  at free boundary point  $x_0 \in \partial\{u > 0\}$  is either identically zero or homogeneous function of degree one. In particular, if  $N = 2$  and  $u$  is not degenerate at  $x_0 \in \partial\{u > 0\}$  then any blow-up of  $u$  at  $x_0$  must be one of the following functions (in appropriate coordinate system):*

- (1)  $\sqrt{2M}x_1^+$ , half plane solution provided that there is a measure theoretic normal at  $x_0$ ,
- (2) wedge  $\alpha|x_1|$ ,  $0 < \alpha \leq \sqrt{2M}$ ,
- (3) two plane solution  $\alpha x_1^+ - \beta x_1^-$ ,  $\alpha^2 - \beta^2 = 2M$ ,  $\alpha, \beta > 0$ .

In order to prove Theorem A we will use a monotonicity formula based on a computation of Joel Spruck [26]. From Theorem A it follows that in  $\mathbb{R}^2$  all blow-up limits at the non-degenerate free boundary points can be explicitly computed. It is worthwhile to note that the minimizers of

$$(1.3) \quad J(u) = \int_{B_1} |\nabla u|^2 + 2M\chi_{\{u>0\}}$$

are non-degenerate, i.e. for each subdomain  $\Omega' \subset\subset B_1$  there is a constant  $c_0 > 0$  depending on  $\text{dist}(\partial B_1, \partial\Omega')$ ,  $N$  such that

$$(1.4) \quad \sup_{B_r(x_0)} u^+ \geq c_0 r, \quad \forall x_0 \in \partial\{u > 0\} \cap \Omega'$$

and  $B_r(x_0) \subset B_1$ . However, if  $u_\varepsilon$  is **any** solution of  $(\mathcal{P}_\varepsilon)$  then non-degeneracy may not be true. There is a sufficient condition [10] Theorem 6.3 that implies (1.4). Another striking difference is that for the stationary case there are wedge-like global solutions for which the measure theoretic free boundary is empty. This is impossible for the minimizers, for the zero set of a minimizer has uniformly positive Lebesgue density. In this respect Theorem A only states that if  $u$  is non-degenerate at  $x_0$  then the blow-up is a nontrivial cone.

The existence of wedge solutions (see Remark 5.1 [10]) suggests that further assumptions are needed to formulate the free boundary condition. One of this is to assume that the positivity set has upper Lebesgue density  $\Theta^*(\{u > 0\}, x) < 1$ , i.e. the upper density measure is not covering the full ball. Observe that for the wedge solution's topological boundary is not the measure theoretic boundary. Our next result addresses the degeneracy and wedge-formation at the free boundary of blow-ups in  $\mathbb{R}^3$ .

**Theorem B.** *Let  $N = 3$  and  $u \geq 0$  such that  $u$  is non-degenerate at  $y_0 \in \partial\{u > 0\}$ . Let  $u_0$  be a blow-up of  $u$  at  $y_0$ . If  $\mathfrak{C}$  is a component of  $\partial\{u_0 > 0\}$  such that the measure theoretic boundary of  $\{u_0 > 0\}$  in  $\mathfrak{C}$  is non-empty then*

- (1) *all points of  $\mathfrak{C}$  are non-degenerate,*
- (2)  *$\mathfrak{C}$  is a subset of the measure theoretic boundary of  $\{u_0 > 0\}$ ,*
- (3)  *$\mathfrak{C} \setminus \{0\}$  is smooth.*

*In particular in  $\mathbb{R}^3$  the singular set of  $\partial\{u_0 > 0\}$  is a singleton.*

We remark that Theorem B implies that the reduced boundary propagates instantaneously in the components of  $\partial\{u_0 > 0\}$ . Our last result establishes a direct connection with capillary surfaces and provides a way to reformulate the classification of blow-ups in terms of capillary surfaces in spheres with contact angle  $\pi/2$ .

**Theorem C.** *Let  $u_0$  be as in Theorem B and  $u_0 = rg(\sigma), \sigma \in \mathbb{S}^2$  in spherical coordinates. Then the parametrization  $\mathcal{X}(\sigma) = \sigma g(\sigma) + \nabla_{\mathbb{S}^2} g(\sigma)$  defines a conformal and minimal immersion. If  $\{g > 0\}$  a disk-type domain then  $u_0$  is a half-plane solution  $\sqrt{2M}x_1^+$ . If  $\{g > 0\}$  is ring-type then the only singular cone is the Alt-Caffarelli catenoid.*

Observe that  $\Delta u_0 = 0$  implies that the spherical part  $g$  satisfies the linear equation on the sphere

$$\Delta_{\mathbb{S}^{N-1}} g + (N-1)g = 0,$$

where  $\Delta_{\mathbb{S}^{N-1}}$  is the Laplace-Beltrami operator. If we regard  $g$  as the support function of some embedded hypersurface then the matrix  $[\nabla_{ij}g + \delta_{ij}g]^{-1}$  gives the Weingarten mapping and its eigenvalues are the principal curvatures  $k_1, \dots, k_{N-1}$ . If  $N = 3$  then we have that

$$0 = \Delta_{\mathbb{S}^2} g + 2g = \text{trace}[\nabla_{ij}g + \delta_{ij}g] = \frac{1}{k_1} + \frac{1}{k_2} = \frac{k_1 + k_2}{k_1 k_2}$$

implying that the mean curvature is zero at the points where the Gauss curvature does not vanish. This is how the minimal surfaces enter into the game. One of the main obstacles is to show that the surface parametrized by  $\mathcal{X}(\sigma) = \nabla u_0(\sigma)$  is embedded. Then the classification for the disk-type  $\{g > 0\}$  follows from a result of Nitsche [19]. To prove the last statement of Theorem C we will use the moving plane method. It is worthwhile to point out that the results of this paper can be extended to other classes of stationary

points. For instance, the weak solutions introduced in [2] can be analyzed in similar way provided that the zero set has uniformly positive Lebesgue density at free boundary points in order to guarantee that the blow-ups of weak solution is again a weak solution, see example 5.8 in [2] and section 5 in [28].

**Related works.** In [13] F. Hélein, L. Hauswirth and F. Pacard have considered the following overdetermined problem

$$(1.5) \quad \begin{cases} \Delta u(x) = 0 & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u(x) = 0, |\nabla u| = 1 & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega$  is a smooth domain and the boundary conditions are satisfied in the classical sense. A domain  $\Omega$  admitting a solution  $u$  to (1.5) is called exceptional. Nonnegative smooth solutions of the limiting singular perturbation problem solve (1.5) with  $M = \frac{1}{2}$ . The authors have constructed a number of examples of exceptional domains in [13] and proposed to classify them. In particular, they proved that if  $\Omega \subset \mathbb{R}^2$  is conformal to half-plane such that  $u$  is strictly monotone in one fixed direction then  $\Omega$  is a half-space, [13, Proposition 6.1]. However the general problem remained open.

Later M. Traizet showed that the smoothness assumption can be relaxed, namely if  $\Omega \subset \mathbb{R}^2$  has  $C^0$  boundary and the boundary conditions are still satisfied in the classical sense then  $\Omega$  is real analytic [27, Proposition 1]. Under various topological conditions on the two dimensional domain  $\Omega \subset \mathbb{R}^2$  (such as finite connectivity and periodicity) M. Traizet classified the possible exceptional domains. One of his remarkable results is that from  $\Omega$  one can construct a *complete* minimal surface using the Weierstrass representation formula [27, Theorem 9]. Another classification result in the two dimensional case under stronger topological hypotheses than in [27] is given by D. Khavinson, E. Lundberg and R. Teodorescu. Moreover, their results in simply connected case are stronger because unlike M. Traizet they do not assume the finite connectivity (i.e.  $\partial\Omega$  has finite number of components). As opposed to these results (1) we do not assume any regularity of the free boundary (which plays the role of  $\partial\Omega$  in (1.5)), (2) the Neumann condition is not satisfied in the classical sense, (3) the minimal surface we construct in Theorem C is *not complete* and it is a capillary surface entrapped in the sphere, and (4) our techniques do not impose any restriction on the dimension. Note that, in [13] the authors suggested to study more general classes of exceptional domains: if  $(M, g)$  is an  $m$ -dimensional Riemannian manifold admitting a harmonic function with zero Dirichlet boundary data and constant Neumann boundary data then  $M$  is called exceptional and  $u$  a roof function. In this context Theorem C provides a way of constructing roof function on the sphere from the blow-ups of stationary points of the Alt-Caffarelli functional.

One may consider higher order critical points as well, such as mountain passes (which are, in fact, minimizers over some subspace) for which one has non-degeneracy and nontrivial Lebesgue density properties [14, Propositions 1.7-5.1]. Observe that neither of these properties is available for our solutions as Theorem 6.3 and Remark 5.1 in [10] indicate, and in the present work we do not impose any additional assumptions on our stationary points of this kind.

The only higher dimensional result that appears in [13], [16] and [27] states that if the complement of  $\Omega$  is connected and has  $C^{2,\alpha}$  boundary, then  $\Omega$  is the exterior of a ball [16, Theorem 7.1]. The restriction  $\Omega \subset \mathbb{R}^2$  is because the authors have mainly used techniques from complex analysis. Our approach does not

have this restriction since our main tool is the representation in terms of the Minkowski support function. We remark that in higher dimensions our construction results a surface in the sphere of radius  $\sqrt{2M}$  such that the sum of its principal radii of curvature is zero, and it is transversal to the sphere.

Finally, we point out that an application of our results is a new proof of the classification of global minimizers in  $\mathbb{R}^3$  [7]. Indeed, Theorem 6 from [22] implies that the capillary surface  $\mathcal{M}$  constructed in Theorem C from the blow-up limit must be totally geodesic (i.e. the second fundamental form is identically zero). Consequently, the blow-up must be the half-plane solution.

## 2. NOTATION

Let us set up notation. Throughout the paper  $N$  will denote the spatial dimension. For any  $x_0 \in \mathbb{R}^N$ ,  $B_r(x_0) = \{x \in \mathbb{R}^N, s.t. |x - x_0| < r\}$ . The  $s$ -dimensional Hausdorff measure is denoted by  $\mathcal{H}^s$ , the unit sphere by  $\mathbb{S}^{N-1} \subset \mathbb{R}^N$ , and the characteristic function of the set  $D$  by  $\chi_D$ . Also

$$M = \int_0^1 \beta(t) dt.$$

Sometimes we will denote  $x = (x_1, x')$  where  $x' \in \mathbb{R}^{N-1}$ . For given function  $v$ , we will denote  $v^+ = \max(0, v)$  and  $v^- = \max(0, -v)$ . Finally, we say that  $v \in C_{loc}^{0,1}(\mathcal{D})$  if for every  $\mathcal{D}' \Subset \mathcal{D}$ , there is a constant  $L(\mathcal{D}')$  such that

$$|v(x) - v(y)| \leq L(\mathcal{D}')|x - y|.$$

If  $v \in C_{loc}^{0,1}(\mathcal{D})$  then we say that  $v$  is locally Lipschitz in  $\mathcal{D}$ . For  $x = (x_1, \dots, x_N)$  and  $x_0 \in \mathbb{R}^N$  fixed we let  $(x - x_0)_1^+$  be the positive part of the first coordinate of  $x - x_0$ .

## 3. MONOTONICITY FORMULA OF SPRUCK

It is convenient to work with a weaker definition of non-degeneracy which only assures that the blow-up does not vanish identically.

**Definition 3.1.** *We say that  $u$  is degenerate at  $x_0$  if  $\liminf_{r \rightarrow 0} \frac{1}{r} \int_{B_r(x_0)} u^+ = 0$ .*

It is known that the solutions of  $(\mathcal{P}_\varepsilon)$  are locally Lipschitz continuous, see Appendix Proposition 7.1. It yields that there is a subsequence  $\varepsilon_j \rightarrow 0$  such that  $u_{\varepsilon_j} \rightarrow u$  locally uniformly, in fact  $u$  is a stationary point of the Alt-Caffarelli problem in some weak sense and the blow-up of  $u$  can be approximated by some scaled family of solutions to  $(\mathcal{P}_\varepsilon)$ , see Appendix.

**Proposition 3.1.** *Let  $u$  be a limit of some sequence  $u_{\varepsilon_j}$  as in Proposition 7.2. Then any blow-up of  $u$  at a non-degenerate point is a homogeneous function of degree one.*

**Proof.** To fix the ideas we assume that  $0 \in \partial\{u > 0\}$  where  $u$  is non-degenerate. We begin with writing the Laplacian in polar coordinates

$$(3.1) \quad \Delta u = u_{rr} + \frac{N-1}{r} u_r + \frac{1}{r^2} \Delta_{\mathbb{S}^{N-1}} u$$

and then introducing the auxiliary function

$$(3.2) \quad v(t, \sigma) = \frac{u(r, \sigma)}{r}, \quad r = e^{-t}.$$

A straightforward computation yields

$$\begin{aligned} v_t &= -u_r + v, \\ v_\sigma &= \frac{u_\sigma}{r}, \\ v_{tt} &= u_{rr}r + v_t, \\ \Delta_{\mathbb{S}^{N-1}} v &= \frac{1}{r} \Delta_{\mathbb{S}^{N-1}} u, \end{aligned}$$

where, with some abuse of notation, we let  $v_\sigma$  denote the gradient on the sphere. Rewriting the equation in  $t$  and  $\sigma$  derivatives we obtain

$$\frac{1}{r}[(N-1)(v - \partial_t v_\varepsilon) + \partial_t^2 v_\varepsilon - \partial_t v_\varepsilon + \Delta_{\mathbb{S}^{N-1}} v_\varepsilon] = \frac{1}{\varepsilon} \beta \left( \frac{r}{\varepsilon} v_\varepsilon \right).$$

Next, we multiply both sides of the last equation by  $\partial_t v_\varepsilon$  to get

$$(3.3) \quad \partial_t v_\varepsilon [(N-1)(v - \partial_t v_\varepsilon) + \partial_t^2 v_\varepsilon - \partial_t v_\varepsilon + \Delta_{\mathbb{S}^{N-1}} v_\varepsilon] = \partial_t v_\varepsilon \frac{r}{\varepsilon} \beta \left( \frac{r}{\varepsilon} v_\varepsilon \right).$$

The right hand side of (3.3) can be further transformed as follows

$$\begin{aligned} \frac{r}{\varepsilon} \beta \left( \frac{e^{-t}}{\varepsilon} v_\varepsilon \right) \partial_t v_\varepsilon &= \beta \left( \frac{e^{-t}}{\varepsilon} v_\varepsilon \right) \left[ \frac{e^{-t}}{\varepsilon} \partial_t v_\varepsilon - \frac{e^{-t}}{\varepsilon} v_\varepsilon \right] + \beta \left( \frac{e^{-t}}{\varepsilon} v_\varepsilon \right) \frac{e^{-t}}{\varepsilon} v_\varepsilon \\ &= \partial_t \int_0^{\frac{e^{-t}}{\varepsilon} v_\varepsilon} \beta(s) ds + \beta \left( \frac{e^{-t}}{\varepsilon} v_\varepsilon \right) \frac{e^{-t}}{\varepsilon} v_\varepsilon \\ &= \partial_t \mathcal{B} \left( \frac{e^{-t}}{\varepsilon} v_\varepsilon \right) + \beta \left( \frac{e^{-t}}{\varepsilon} v_\varepsilon \right) \frac{e^{-t}}{\varepsilon} v_\varepsilon \\ &\equiv I_1. \end{aligned}$$

It is important to note that by our assumption (1.1) the last term is nonnegative, in other words

$$(3.4) \quad \beta \left( \frac{e^{-t}}{\varepsilon} v_\varepsilon \right) \frac{e^{-t}}{\varepsilon} v_\varepsilon \geq 0.$$

Moreover, we have

$$\begin{aligned} I_2 &\equiv [(N-1)v_\varepsilon - N\partial_t v_\varepsilon + \partial_t^2 v_\varepsilon + \Delta_{\mathbb{S}^{N-1}} v_\varepsilon] \partial_t v_\varepsilon \\ &= (N-1)\partial_t \left( \frac{v_\varepsilon^2}{2} \right) - Nv_t^2 + \partial_t \left( \frac{(\partial_t v_\varepsilon)^2}{2} \right) + \partial_t v_\varepsilon \Delta_{\mathbb{S}^{N-1}} v_\varepsilon. \end{aligned}$$

Next we integrate the identity

$$I_2 = rI_1$$

over  $\mathbb{S}^{N-1}$  and then over  $[T_0, T]$  to get

$$\begin{aligned} (N-1) \int_{\mathbb{S}^{N-1}} \frac{v_\varepsilon^2}{2} \Big|_{T_0}^T - N \int_{T_0}^T \int_{\mathbb{S}^{N-1}} (\partial_t v_\varepsilon)^2 + \int_{\mathbb{S}^{N-1}} \frac{(\partial_t v_\varepsilon)^2}{2} \Big|_{T_0}^T + \int_{T_0}^T \int_{\mathbb{S}^{N-1}} \partial_t v_\varepsilon \Delta_{\mathbb{S}^{N-1}} v_\varepsilon \\ = \int_{\mathbb{S}^{N-1}} \mathcal{B} \left( \frac{e^{-t}}{\varepsilon} v_\varepsilon \right) \Big|_{T_0}^T + \int_{T_0}^T \int_{\mathbb{S}^{N-1}} \beta \left( \frac{r}{\varepsilon} v_\varepsilon \right) \frac{r}{\varepsilon} v_\varepsilon. \end{aligned}$$

Note that

$$(3.5) \quad \int_{T_0}^T \int_{\mathbb{S}^{N-1}} \partial_t v_\varepsilon \triangle_{\mathbb{S}^{N-1}} v_\varepsilon = -\frac{1}{2} \int_{\mathbb{S}^{N-1}} |\nabla_\sigma v_\varepsilon|^2 \Big|_{T_0}^T.$$

Rearranging the terms and utilizing (3.4) we get the identity

$$(3.6) \quad \begin{aligned} N \int_{T_0}^T \int_{\mathbb{S}^{N-1}} (\partial_t v_\varepsilon)^2 + \int_{T_0}^T \int_{\mathbb{S}^{N-1}} \beta \left( \frac{r}{\varepsilon} v_\varepsilon \right) \frac{r}{\varepsilon} v_\varepsilon &= (N-1) \int_{\mathbb{S}^{N-1}} \frac{v_\varepsilon^2}{2} \Big|_{T_0}^T + \int_{\mathbb{S}^{N-1}} \frac{(\partial_t v_\varepsilon)^2}{2} \Big|_{T_0}^T \\ &\quad - \frac{1}{2} \int_{\mathbb{S}^{N-1}} |\nabla_\sigma v_\varepsilon|^2 \Big|_{T_0}^T - \int_{\mathbb{S}^{N-1}} \mathcal{B} \left( \frac{e^{-t}}{\varepsilon} v_\varepsilon \right) \Big|_{T_0}^T. \end{aligned}$$

Therefore we infer the inequality

$$(3.7) \quad \int_{T_0}^T \int_{\mathbb{S}^{N-1}} (\partial_t v_\varepsilon)^2 \leq C$$

where  $C$  depends on  $\|\nabla u_\varepsilon\|_\infty, M, N$  but not on  $\varepsilon, T_0$  or  $T$ .

Letting  $\varepsilon \rightarrow 0$  we conclude

$$(3.8) \quad \int_{T_0}^T \int_{\mathbb{S}^{N-1}} (\partial_t v)^2 \leq C$$

where  $v(t, \sigma) = \frac{u(r, \sigma)}{r}$ . But  $\partial_t v = -u_r + \frac{u}{r}$  thus implying that

$$(3.9) \quad \int_{T_0}^\infty \int_{\mathbb{S}^{N-1}} \left( u_r - \frac{u}{r} \right)^2 dt d\sigma \leq C.$$

The proof of Theorem A follows if we note that  $-u_r + \frac{u}{r} = 0$  is the Euler equation for homogeneous functions of degree one.  $\square$

In the proof of Proposition 3.1 we used Spruck's original computation [26]. The previous result can be reformulated as follows.

**Corollary 3.2.** *Let  $(r, \sigma), \sigma \in \mathbb{S}^{N-1}$  be the spherical coordinates and set*

$$(3.10) \quad S_\varepsilon(r) = \int_{\mathbb{S}^{N-1}} \left\{ 2\mathcal{B}(u_\varepsilon(r, \sigma)/\varepsilon) + \frac{1}{r^2} |\nabla_\sigma u_\varepsilon|^2 - (N-1) \frac{u_\varepsilon^2(r, \sigma)}{r^2} - \left( \partial_r u_\varepsilon(r, \sigma) - \frac{u_\varepsilon(r, \sigma)}{r} \right)^2 \right\} d\sigma.$$

- Then  $S_\varepsilon(r)$  is nondecreasing in  $r$ .
- Moreover, if  $u_{\varepsilon_j} \rightarrow u$  for some subsequence  $\varepsilon_j \rightarrow 0$ , then  $S_{\varepsilon_j}(r) \rightarrow S(r)$  for a.e.  $r$  where

$$(3.11) \quad S(r) = \int_{\mathbb{S}^{N-1}} \left\{ 2M\chi_{\{u>0\}} + \frac{1}{r^2} |\nabla_\sigma u|^2 - (N-1) \frac{u^2(r, \sigma)}{r^2} - \left( \partial_r u(r, \sigma) - \frac{u(r, \sigma)}{r} \right)^2 \right\} d\sigma.$$

Furthermore,  $S(r)$  is nondecreasing function of  $r$ ,

- $S(r)$  is constant if and only if  $u$  is a homogenous function of degree one.

**Proof.** By setting  $r_1 = e^{-T}$ ,  $r_2 = e^{-T_0}$  and noting that  $r_1 < r_2$  if  $T > T_0$  we obtain from (3.6)

$$S_\varepsilon(r_2) - S_\varepsilon(r_1) = 2N \int_{T_0}^T \int_{\mathbb{S}^{N-1}} (\partial_t v_\varepsilon)^2 + 2 \int_{T_0}^T \int_{\mathbb{S}^{N-1}} \beta\left(\frac{r}{\varepsilon} v_\varepsilon\right) \frac{r}{\varepsilon} v_\varepsilon \geq 0$$

where we applied (3.4) and the first part follows. The second part follows from Propositions 7.1 and 7.2. Indeed, integrating  $S_\varepsilon(r) \leq S_\varepsilon(r+t)$ ,  $t \geq 0$  over  $[r_1 - \delta, r_1 + \delta]$  we infer

$$\frac{1}{2\delta} \int_{r_1-\delta}^{r_1+\delta} S_\varepsilon(r) dr \leq \frac{1}{2\delta} \int_{r_1-\delta}^{r_1+\delta} S_\varepsilon(r+t) dr.$$

Then first letting  $\varepsilon \rightarrow 0$  and utilizing Proposition 7.2 together with (7.1) and then sending  $\delta \rightarrow 0$  we infer that  $S(r)$  is nondecreasing for a.e.  $r$ . Finally the last part follows as in the proof of Proposition 3.1.  $\square$

As one can see we did not use Pohozaev type identity as opposed to the monotonicity formula in [28]. Spruck's monotonicity formula enjoys a remarkable property.

**Lemma 3.3.** *Let  $u$  be as in Proposition 3.1. Set  $S(x_0, r, u)$  to be  $S(r)$  corresponding the sphere centered at  $x_0 \in \partial\{u > 0\}$ . Suppose  $x_k \in \partial\{u > 0\}$  such that  $x_k \rightarrow x_0$  then*

$$\limsup_{x_k \rightarrow x_0} S(x_k, 0, u) \leq S(x_0, 0, u).$$

**Proof.** For given  $\delta > 0$  there is  $\rho_0 > 0$  such that  $S(x_0, \rho, u) \leq S(x_0, 0, u) + \delta$  whenever  $\rho < \rho_0$ . Fix such small  $\rho$  and choose  $k$  so large that  $S(x_k, \rho, u) < \delta + S(x_0, \rho, u)$ . From the monotonicity of  $S(x_k, \rho, u)$  and above estimates we have

$$\begin{aligned} S(x_k, 0, u) &\leq S(x_k, \rho, u) \leq \delta + S(x_0, \rho, u) \\ &\leq 2\delta + S(x_0, 0, u). \end{aligned}$$

First letting  $x_k \rightarrow x_0$  and then  $\delta \rightarrow 0$  the result follows.  $\square$

#### 4. PROOF OF THEOREM A

The first part of the theorem follows from Proposition 3.1. Since  $u$  is not degenerate at the origin then by Propositions 7.2 and 7.5  $u_{\rho_k}(x) \rightarrow u_0(x)$  locally uniformly and by Proposition 3.1  $u_0$  is homogeneous of degree one. Write  $\triangle$  in polar coordinates  $(r, \theta)$ , and we obtain that

$$\triangle w = \frac{1}{r} \frac{\partial}{\partial r} (r w_r) + \frac{1}{r^2} \frac{\partial}{\partial \theta} (w_\theta).$$

In particular, writing  $u_0 = r g(\theta)$ , this yields a second order ODE for  $g$

$$(4.1) \quad g + \ddot{g} = 0.$$

Suppose  $g(0) = g(\theta_0) = 0$ ,  $\theta_0 \in [0, 2\pi)$ , then (4.1) implies that  $g(\theta) = A \sin \theta$  for some constant  $A$ , and consequently forcing  $\theta_0 = \pi$ . Hence, since  $N = 2$ , we obtain that  $u_0$  must be linear, in other words the free boundary  $\partial\{u_0 > 0\}$  is everywhere flat. This in turn implies that in two dimensions the singular set of the free boundary  $\partial\{u_0 > 0\}$  is empty. Consequently,  $u_0$  is linear in  $\{u_0 > 0\}$  and  $\{u_0 < 0\}$ . From here the parts (2) and (3) of Theorem A follow from Propositions 5.3 and 5.1 of [10].



So it remains to check (1). The only difference from the parabolic setting of [10] is that for elliptic case the limit function  $M(x) = \lim_{\delta_j \rightarrow 0} \mathcal{B}_{\delta_j}(u_{\delta_j})$  cannot have nontrivial concentration on the free boundary coming from  $\{x_1 < 0\}$ . By Proposition 7.5 there is sequence  $0 < \lambda_j \rightarrow 0$  such that  $(u_{\varepsilon_j})_{\lambda_j} \rightarrow u_0, \varepsilon_j/\lambda_j \rightarrow 0$  and  $M(x) = M\chi_{\{x_1 > 0\}} + M_0\chi_{\{x_1 < 0\}}$ . It follows from (7.2) that

$$(4.2) \quad \int_{\{x_1 > 0\}} M \partial_1 \phi + \int_{\{x_1 < 0\}} M_0 \partial_1 \phi = \int_{\{x_1 > 0\}} \frac{\alpha^2}{2} \partial_1 \phi.$$

Next we claim that  $M_0 = 0$ . To see this we take  $\mathcal{C}_\gamma = \{x \in \mathcal{C} : \gamma < \mathcal{B}(u_{\delta_j}/\delta_j) < M - \gamma\}$  where  $\delta_j = \varepsilon_j/\lambda_j \rightarrow 0$ . There is  $\gamma' > 0$  such that  $\mathcal{C}_\gamma \subset \{\gamma' < \frac{u_{\delta_j}}{\delta_j} < 1 - \gamma'\}$  and  $|\{\gamma' < \frac{u_{\delta_j}}{\delta_j} < 1 - \gamma'\}| \rightarrow 0$  because  $u_{\delta_j} \rightarrow u_0$ . Thus  $\gamma < M_0 < M - \gamma$ . Since  $\gamma$  is arbitrary positive number it follows that  $M_0 = 0$ . Then (4.2) yields  $\alpha = \sqrt{2M}$ .  $\square$

## 5. STRUCTURE OF THE FREE BOUNDARY OF BLOW-UPS IN $\mathbb{R}^3$

In this section we assume that  $u \geq 0$  is a limit of  $u_{\varepsilon_j}$  solving  $(\mathcal{P}_\varepsilon)$  for some sequence  $\varepsilon_j \rightarrow 0$ ,  $u$  is non-degenerate at  $y_0 \in \partial\{u > 0\}$  and  $u_0$  is a blow-up of  $u$  at 0. Note that by Corollary 3.2  $u_0$  is homogenous function of degree one. If  $u_0$  is not a minimizer then it is natural to expect that the solutions of  $(\mathcal{P}_\varepsilon)$  develop singularities in  $\mathbb{R}^N, N \geq 3$ .

**5.1. Properties of  $\partial\{u_0 > 0\}$ .** We want to study the properties of  $g$ . We first prove a Bernstein-type result which is a simple consequence of a refinement of Alt-Caffarelli-Friedman monotonicity formula [8].

**Lemma 5.1.** *Let  $u \geq 0$  be a limit of solutions to  $(\mathcal{P}_\varepsilon)$  and  $u_0 = rg(\sigma), \sigma \in \mathbb{S}^{N-1}$  be a nontrivial blow-up of  $u$  at some free boundary point. If there is a hemisphere containing  $\text{supp } g$  then  $u_0$  is a half-plane.*

**Proof.** Without loss of generality we assume that  $\text{supp } g \subset \mathbb{S}_+^{N-1} = \{X \in \mathbb{S}^{N-1}, x_N \geq 0\}$ . Let  $v(x_1, \dots, x_N) = u(x_1, \dots, -x_N)$  be the reflection of  $u$  with respect to the hyperplane  $x_N = 0$ . Then  $v$  is nonnegative subharmonic function satisfying the requirements of Lemma 2.3 [8]. Thus

$$\Phi(r) = \frac{1}{r^4} \int_{B_r} \frac{|\nabla u_0|^2}{|x|^{N-2}} \int_{B_r} \frac{|\nabla v|^2}{|x|^{N-2}}$$

is nondecreasing in  $r$ . Moreover

$$\Phi'(r) \geq \frac{2\Phi(r)}{r} A_r, \quad A_r = \frac{C_N}{r^{N-1}} \text{Area}(\partial B_r \setminus (\text{supp } u_0 \cup \text{supp } v)).$$

Thus, if  $\text{supp } g$  digresses from the hemisphere by size  $\delta > 0$  then integrating the differential inequality for  $\Phi$  we see that  $\Phi$  grows exponentially which is a contradiction, since in view of Proposition 7.3  $u_0$  is Lipschitz and hence  $\Phi$  must be bounded.  $\square$

It is convenient to define the following subsets of the free boundary

$$(5.1) \quad \Gamma_{\frac{1}{2}} = \left\{ x \in \partial\{u_0 > 0\} \text{ s.t. } \Theta(x, \{u_0 > 0\}) = \frac{1}{2} \right\},$$

$$(5.2) \quad \Gamma_1 = \{x \in \partial\{u_0 > 0\} \text{ s.t. } \Theta(x, \{u_0 > 0\}) = 1\}.$$

Here  $\Theta(x, D)$  is the Lebesgue density of  $D$  at  $x$ . We will see that  $\Theta(x, \{u_0 > 0\})$  exists at each non-degenerate point and equals either 1 or  $\frac{1}{2}$ .

**Lemma 5.2.** *Let  $x_0 \in \partial\{u_0 > 0\} \setminus \{0\}$  be non-degenerate free boundary point such that the lower Lebesgue density  $\Theta_*(x_0, \{u_0 \equiv 0\}) > 0$ , and  $N = 3$ . Then there is a unit vector  $\nu_0$  such that*

$$(5.3) \quad u_0(x) = \sqrt{2M}[(x - x_0) \cdot \nu_0]^+ + o(x - x_0).$$

*In particular,  $x_0 \in \Gamma_{\frac{1}{2}}$*

**Proof.** Set  $v_k = \frac{u_0(x_0 + \rho_k x)}{\rho_k}$ , since  $u_0$  is non-degenerate at  $x_0$  it follows from a customary compactness argument that  $v_k \rightarrow v$  and from Corollary 3.2  $v$  is homogeneous function of degree one. We have

$$(5.4) \quad \frac{u_0(x_0 + \rho_k x)}{\rho_k} = u_0(\rho_k^{-1}x_0 + x) = \nabla u_0(\rho_k^{-1}x_0 + x)(\rho_k^{-1}x_0 + x) =$$

$$(5.5) \quad = \frac{1}{\rho_k} \nabla u_0(x_0 + \rho_k x)(x_0 + \rho_k x)$$

where the last line follows from the zero degree homogeneity of the gradient, hence

$$\rho_k v_k(x) = \nabla u_0(x_0 + \rho_k x)(x_0 + \rho_k x) = \nabla v_k(x)(x_0 + \rho_k x).$$

By Lipschitz continuity of  $u_0$  it follows that  $v_k$  is locally bounded. Consequently, for a suitable subsequence of  $\rho_k$  we have that  $v_{k_j} \rightarrow v$  and  $\nabla v(x)x_0 = 0$ . Without loss of generality we may assume that  $x_0$  is on the  $x_3$  axis, implying that  $v$  depends only on  $x_1$  and  $x_2$ . Applying Proposition 7.6 and Corollary 3.2 we conclude that  $S(x_0, r, u_0)$  is non-decreasing and thus  $v$  must be homogeneous of degree one. Indeed, there is a sequence  $\delta_j \rightarrow 0$  such that  $(u_{\varepsilon_j})_{\lambda_j} \rightarrow v$ ,  $\delta_j = \varepsilon_j/\lambda_j$  by Proposition 7.6.

Finally, applying Theorem A and the assumption  $\Theta_*(x_0, \{u_0 = 0\}) > 0$  we see that  $v$  must be a half-plane solution. It remains to note that the approximate tangent of  $\partial\{u_0 > 0\}$  at  $x_0$  is unique and this completes the proof.  $\square$

**Lemma 5.3.** *Let  $u_0$  be a nontrivial blow-up of  $u$  and  $N = 3$ . Then the set of degenerate points of  $\partial\{u_0 > 0\}$  is empty.*

**Proof.** Let  $u_0$  be a blow-up of  $u$  at 0. Since  $u$  is non-degenerate at 0 then it follows that  $u_0$  does not vanish identically. Hence there is a ball  $B \subset \{u_0 > 0\}$  touching  $\partial\{u_0 > 0\}$  at some point  $z_0 \in \partial\{u_0 > 0\} \cap B$ . By Hopf's lemma, Lipschitz estimate 7.3 (i) and asymptotic expansion [5] Lemma A1 it follows that  $u_0$  is not degenerate at  $z_0$ . Consequently, the set of non-degenerate points of  $u_0$  is not empty.

Suppose that  $u_0$  is degenerate at  $y_0 \in \partial\{u_0 > 0\} \setminus \{0\}$ , then there is a neighbourhood of  $y_0$  where at all points  $u_0$  is degenerate. Indeed, if the claim fails then there is a sequence  $\{y_k\}$  such that  $\partial\{u_0 > 0\} \ni y_k \rightarrow y_0$  and  $u_0$  is non-degenerate at  $y_k$ . Let  $u_{00}^k$  be a blow-up of  $u_0$  at  $y_k$ . By Proposition 7.6 for fixed  $k$  there are  $\delta_j^k \rightarrow 0$  such that  $(u_{\varepsilon_j^k})_{\lambda_j^k} \rightarrow u_{00}^k$ ,  $\delta_j^k = \varepsilon_j^k/\lambda_j^k$ . Thus applying Theorem A it follows that  $u_{00}^k$  is a half plane solution or a wedge.

From scaling properties of Spruk's monotonicity formula we get

$$S(0, y_k, u_0) = S(1, 0, u_{00}^k) = 2M \text{Area}(\mathbb{S}^2 \cap \{u_{00}^k > 0\}) \geq M \text{Area}(\mathbb{S}^2).$$

Then applying Corollary 3.2 to  $u_{\delta_j^k}$  and using the semicontinuity of  $S$  Lemma 3.3 we have

$$M \text{Area}(\mathbb{S}^2) \leq \limsup_{y_k \rightarrow y_0} S(0, y_k, u_0) \leq S(0, y_0, u_0) = 0$$

which is a contradiction. Thus,  $u_0$  is degenerate in some neighbourhood of  $y_0$ .

Case 1: There is  $r > 0$  such that  $|\{u_0 \equiv 0\} \cap B_r(y_0)| > 0$ . Since  $u_0$  is harmonic in  $\{u_0 > 0\}$  and vanishes of first order on  $\partial\{u_0 > 0\}$  then it follows that  $u_0$  is harmonic in  $B_r(y_0)$  hence identically zero implying that  $y_0 \notin \partial\{u > 0\}$  which is a contradiction.

Case 2: There is  $r > 0$  such that  $|\{u_0 \equiv 0\} \cap B_r(y_0)| = 0$ . Then by non-degeneracy theorem 6.3 [10] it follows that  $u_0$  is not degenerate at  $y_0$  which is a contradiction.

Thus  $u_0$  is non-degenerate at every point of  $\partial\{u_0 > 0\} \setminus \{0\}$ .  $\square$

After ruling out the existence of degenerate points we next have to eliminate the wedge-like solutions and show that  $\Gamma_{\frac{1}{2}}$  (which coincides with the reduced boundary) instantaneously propagates in  $\partial\{u_0 > 0\}$ .

**Lemma 5.4.** *Let  $u_0$  be as in Lemma 5.3. Then the following two statements hold*

- (i)  $\Gamma_1$  cannot have limit points in  $\Gamma_{\frac{1}{2}}$ ,
- (ii)  $\partial\{u_0 > 0\} = \Gamma_1 \cup \Gamma_{\frac{1}{2}}$  and consequently  $\Gamma_{\frac{1}{2}}$  is open

**Proof.** Suppose  $x_k \in \Gamma_1$  and  $x_k \rightarrow x \in \Gamma_{\frac{1}{2}}$ . From the semicontinuity of Spruck's energy, Lemma 3.3 we have

$$2M\text{Area}(\mathbb{S}^2) = \limsup_{x_k \rightarrow x} S(x_k, 0, u_0) \leq S(x, 0, u_0) = M\text{Area}(\mathbb{S}^2)$$

which is a contradiction. Thus  $\Gamma_1$  is closed in  $\partial\{u_0 > 0\}$ . By dimension reduction as in the proof of Lemma 5.2, Theorem A and Lemma 5.3 it follows that  $\partial\{u_0 > 0\} = \Gamma_1 \cup \Gamma_{\frac{1}{2}}$ . Therefore  $\Gamma_{\frac{1}{2}}$  is open.  $\square$

## 5.2. Properties of $\Gamma_{\frac{1}{2}}$ .

**Lemma 5.5.** *Suppose  $u_0$  is not degenerate at  $x_0 \in \partial\{u_0 > 0\} \setminus \{0\}$ , such that  $\Theta_*(x_0, \{u = 0\}) > 0$ . Then there is a unique component  $\mathfrak{C}$  of  $\partial\{u_0 > 0\}$  containing  $x_0$  such that  $\mathfrak{C} \subset \Gamma_{\frac{1}{2}}$ .*

**Proof.** We only have to show the uniqueness of  $\mathfrak{C}$ , the rest follows from Lemma 5.2. Suppose, there are two components of  $\partial\{u_0 > 0\} \setminus \{0\}$ ,  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$ , containing  $x_0$ . From the dimension reduction argument as in the proof of Lemma 5.2 it follows that  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$  have the same approximate tangent plane at  $x_0$ . This is in contradiction with our assumption  $\Theta_*(x_0, \{u = 0\}) > 0$ . Note that by Lemma 5.4 the two components  $\mathfrak{C}_1$  and  $\mathfrak{C}_2$  cannot bound components of  $\{u_0 > 0\}$  (lying on the "same side") with non-empty intersection.  $\square$

**Lemma 5.6.** *Let  $u_0$  be as above. Then*

- (i)  $\mathcal{H}^2(\Gamma_{\frac{1}{2}} \cap B_R) < \infty$ , for any  $R > 0$ ,
- (ii) away from  $\Gamma_1$  the representation formula holds

$$\Delta u_0 = \sqrt{2M}\mathcal{H}^2 \llcorner \Gamma_{\frac{1}{2}}.$$

**Proof.** (i) For given  $x \in \Gamma_{\frac{1}{2}}$  there is a  $\tilde{\rho}_x > 0$  such that

$$(5.6) \quad \sup_{B_r(x)} u_0 \geq \sqrt{M}r, \quad r \in (0, \tilde{\rho}_x).$$

This follows from the asymptotic expansion in Lemma 5.2. Consequently, there is  $\rho'_x > 0$  such that

$$(5.7) \quad \int_{B_r(x)} \Delta u_0 \geq \sqrt{M}r^2, \quad r \in (0, \rho'_x).$$

Indeed, if this inequality is false then there is a sequence  $r_j \searrow 0$  such that

$$\int_{B_{r_j}(x)} \Delta u_0 < \sqrt{M}r_j^2.$$

Set  $v_j(x) = \frac{u_0(x+r_jx)}{r_j}$ . By (5.6)  $\sup_{B_1} v_j(x) \geq \sqrt{M}$ . Moreover, it follows from Lemma 5.2 that  $v_j(x) \rightarrow \sqrt{2M}x_1^+$  in a suitable coordinate system, while  $\int_{B_1} \Delta v \leq \sqrt{M}$ . However,  $\int_{B_1} \Delta x_1^+ = \sqrt{2M}\frac{\pi}{2}$  and this is in contradiction with the former inequality. Putting  $\bar{\rho}_x = \min(\rho'_x, \tilde{\rho}_x)$  we see that the collection of balls  $\mathcal{F} = \bigcup B_{\rho_x}(x), x \in \Gamma_{\frac{1}{2}} \cap B_R, \rho_x < \bar{\rho}_x$  is a Besicovitch type covering of  $\Gamma_{\frac{1}{2}} \cap B_R$ . Consequently, there is a positive integer  $m > 0$  and subcoverings  $\mathcal{F}_1, \dots, \mathcal{F}_m$  such that the balls in each of  $\mathcal{F}_i, 1 \leq i \leq m$  are disjoint and  $\Gamma_{\frac{1}{2}} \cap B_R \subset \bigcup_{i=1}^m \mathcal{F}_i$ . We have from (5.7)

$$4\pi R^2 \|\nabla u_0\|_\infty \geq \int_{\partial B_R} \partial_\nu u_0 \geq \int_{B_{\rho_x}(x) \in \mathcal{F}_i} \Delta u_0 = \sum_{B_{\rho_x}(x) \in \mathcal{F}_i} \int_{B_{\rho_x}(x)} \Delta u_0 \geq m\sqrt{M} \sum_{B_{\rho_x}(x) \in \mathcal{F}_i} \rho_x^2.$$

This yields

$$(5.8) \quad \sum_{B_{\rho_x}(x) \in \bigcup_{i=1}^m \mathcal{F}_i} \rho_x^2 \leq \frac{4\|\nabla u_0\|_\infty \pi R^2}{m\sqrt{M}}.$$

Given  $\delta > 0$  small, suppose there is  $x \in \Gamma_{\frac{1}{2}}$  such that  $\bar{\rho}_x \geq \delta$ . Then we choose  $\rho_x < \delta$ . Thus, in any case we can assume that  $\rho_x < \delta$ . In view of (5.8) this implies that the  $\delta$ -Hausdorff premeasure is bounded independently of  $\delta$ . This proves (i).

(ii) From the estimate

$$\sqrt{M}r^2 \leq \int_{B_r(x)} \Delta u_0 \leq 4\pi r^2 \|\nabla u_0\|, \quad r \in (0, \bar{\rho}_x), B_r(x) \cap \Gamma_{\frac{1}{2}} \subset \Gamma_{\frac{1}{2}}$$

we see that there is a positive bounded function  $q$  such that  $\Delta u_0 = q\mathcal{H}^2 \llcorner \Gamma_{\frac{1}{2}}$ . Using Lemma 5.2 we conclude that  $q = \sqrt{2M}$ .  $\square$

Next we prove the full non-degeneracy of  $u_0$  near  $\Gamma_{\frac{1}{2}}$ .

**Lemma 5.7.** *Let  $u_0$  be as above and  $x_0 \in \Gamma_{\frac{1}{2}}$  then for any  $B_r(x)$  such that  $x \in \partial\{u_0 > 0\}$ ,  $B_r(x) \cap \partial\{u_0 > 0\} \subset \Gamma_{\frac{1}{2}}$  we have*

$$\sup_{B_r(x)} u_0 \geq \sqrt{2M}\pi r.$$

**Proof.** By a direct computation we have

$$r^{-2} \int_{\partial B_r(x)} u_0 = \int_0^r \frac{dt}{t^2} \int_{B_t(x)} \Delta u_0 \geq \int_0^r \sqrt{2M} \pi t^2 = \sqrt{2M} \pi r$$

where the inequality follows from the representation formula and the fact that  $\partial\{u_0 > 0\}$  is a cone, and hence it contains a diameter of  $\partial B_t(x)$  for all  $t \in (0, r)$ . It remains to note that  $r^{-2} \int_{\partial B_r(x)} u_0 \leq \sup_{B_r(x)} u_0$ .  $\square$

**5.3. Weak solutions of Alt-Caffarelli problem.** Combining Lemmas 5.6 and 5.7 as well as Propositions 7.2 (iii) and 7.3 (i) we see that  $u_0$  is a weak solution near  $\Gamma_{\frac{1}{2}}$  in the sense of Definition 5.1 [2]. Furthermore,  $\partial\{u_0 > 0\} \setminus \{0\}$  is flat at each point.

**Lemma 5.8.**  $u_0$  is a weak solution in Alt-Caffarelli sense away from  $\Gamma_1$  and  $\Gamma_{\frac{1}{2}}$  is smooth.

**Proof.** All conditions in Definition 5.1 [2] are satisfied and  $u_0$  is flat at every point  $z_0 \in \partial\{u_0 > 0\} \setminus \{0\}$  thanks to (5.3). Applying Theorem 8.1 [2] we infer that  $\Gamma_{\frac{1}{2}}$  is smooth at every  $z_0 \in \partial\{u_0 > 0\} \setminus \{0\}$ .  $\square$

**5.4. Minimal perimeter.** In this section we prove that the perturbations  $S' \subset \{u_0 > 0\}$  of a portion  $S \subset \Gamma_{\frac{1}{2}}$  has larger  $\mathcal{H}^2$  measure than  $S$ . This can be seen from the estimate  $|\nabla u_0(x)| \leq \sqrt{2M}$  which follows from Lemma 7.7. Since by Lemma 5.8 on  $\Gamma_{\frac{1}{2}}$  the free boundary condition  $|\nabla u_0| = \sqrt{2M}$  is satisfied in the classical sense, it follows that

$$\begin{aligned} 0 &= \int_D \Delta u_0 = \int_S \partial_\nu u_0 + \int_{S'} \partial_\nu u_0 = \\ &= \sqrt{2M} \mathcal{H}^2(S) + \int_{S'} \partial_\nu u_0. \end{aligned}$$

But  $|\int_{S'} \partial_\nu u_0| \leq \sqrt{2M} \mathcal{H}^2(S')$  and thereby

$$(5.9) \quad \mathcal{H}^2(S) \leq \mathcal{H}^2(S').$$

The estimate for the perimeter can be reformulated as follows:

**Theorem 5.9.** Let  $N = 3$ , then the components of  $\Gamma_{\frac{1}{2}}$  are surfaces of non-positive outward mean curvature. In particular,  $\Gamma_{\frac{1}{2}}$  is a union of smooth convex surfaces.

**Proof.** Since  $u_0$  is a weak solution then by Lemma 5.8  $\Gamma_{\frac{1}{2}}$  is smooth. If  $z_0 \in \Gamma_{\frac{1}{2}}$  then choosing the coordinate system in  $\mathbb{R}^3$  so that  $x_3$ -axis has the direction of the inward normal of  $\{u_0 > 0\}$  at  $z_0$  and considering the free boundary near  $z_0$  as a graph  $x_3 = h(x_1, x_2)$  we can consider the one-sided variations of the surface area functional. Indeed, let  $\mathcal{D} \subset \mathbb{R}^2$  be a open bounded domain in  $x_1 x_2$  plane containing  $z_0$  and assume  $t > 0, 0 \leq \psi \in C_0^\infty(\mathcal{D})$ . Then from (5.9) we have

$$\begin{aligned} (5.10) \quad 0 &\geq \frac{1}{t} \int_{\mathcal{D}} [\sqrt{1 + |\nabla h|^2} - \sqrt{1 + |\nabla(h - t\psi)|^2}] = \\ &= \int_{\mathcal{D}} \frac{2\nabla h \nabla \psi - t|\nabla \psi|^2}{\sqrt{1 + |\nabla h|^2} + \sqrt{1 + |\nabla(h - t\psi)|^2}} \rightarrow \text{ as } t \rightarrow 0 \\ &\rightarrow \int_{\mathcal{D}} \frac{\nabla h \nabla \psi}{\sqrt{1 + |\nabla h|^2}}. \end{aligned}$$

Therefore  $\operatorname{div} \left( \frac{\nabla h}{\sqrt{1+|\nabla h|^2}} \right) \geq 0$  and noting that  $\Gamma_{\frac{1}{2}}$  is a cone the result follows.  $\square$

**Lemma 5.10.** *Let  $\mathfrak{C}$  be a component of  $\partial\{u_0 > 0\}$  such that  $\mathfrak{C} \cap \Gamma_{\frac{1}{2}} \neq \emptyset$ . Then  $\mathfrak{C} \setminus \Gamma_{\frac{1}{2}} = \emptyset$ , in other words all points of  $\mathfrak{C}$  are in  $\Gamma_{\frac{1}{2}}$ .*

**Proof.** By Lemma 5.3  $\mathfrak{C}$  cannot have degenerate points, thus we have to show that  $\Gamma_{\frac{1}{2}}$  cannot have limit points in  $\Gamma_1$ . Note that  $\Gamma_{\frac{1}{2}}$  is of locally finite perimeter (see Lemma 5.6 (i)) and hence locally it is a countable union of convex surfaces. Let  $x_0 \in \Gamma_1 \cap \mathfrak{C}$  be a limit point of  $\Gamma_{\frac{1}{2}} \cap \mathfrak{C}$ . The generatrix of the cone  $\partial\{u_0 > 0\}$  passing through  $x_0$  splits  $\mathfrak{C}$  into two parts one of which must be convex near  $x_0$  because by assumption  $x_0$  is a limit point of  $\Gamma_{\frac{1}{2}}$ , see Theorem 5.9. The set  $\{u_0 = 0\}^\circ$  propagates to  $x_0$  because  $\Gamma_{\frac{1}{2}}$  is a subset of reduced boundary. Thus, there is another subset of  $\Gamma_{\frac{1}{2}}$  approaching to  $x_0$ , and it is a part of the topological boundary of  $\{u_0 = 0\}^\circ$ . Therefore, the ray passing through  $x_0$  is on the boundaries of two convex pieces of  $\partial\{u_0 > 0\}$  (near  $x_0$ ). Note that if these pieces of  $\Gamma_{\frac{1}{2}}$  contain flat parts then from the unique continuation theorem we infer that  $\partial\{u_0 > 0\}$  cannot have singularity at 0. Thus, they cannot contain flat parts and consequently the density of  $\{u_0 > 0\}$  at  $x_0$  cannot be 1, because by convexity of  $\Gamma_{\frac{1}{2}}$  it follows that  $\{u_0 \equiv 0\}^\circ$  has positive density at  $x_0$ . But this is in contradiction with the assumption  $x_0 \in \Gamma_1$ .  $\square$

We summarize our analysis above by stating the following

**Proposition 5.11.** *Let  $u_0$  be as above and  $N = 3$ , then  $\partial\{u_0 > 0\} \setminus \{0\}$  is a union of smooth convex cones.*

**5.5. Proof of Theorem B.** The first part of Theorem B follows from Lemma 5.3 while the second part is a corollary of Lemma 5.10 since  $\Gamma_{\frac{1}{2}}$  coincides with the reduced boundary. Finally, the last part follows from Lemma 5.8, because by Lemma 5.10 the reduced boundary propagates instantaneously in the components of  $\partial\{u_0 > 0\}$ .

## 6. PROOF OF THEOREM C

**6.1. Inverse Gauss map and the support function.** Suppose  $u = rg(\theta, \phi)$ ,  $N = 3$  and write the Laplacian in polar coordinates

$$\Delta u_0 = \frac{1}{r} \left( g_{\theta\theta} + \frac{\cos \theta}{\sin \theta} g_\theta + \frac{g_{\phi\phi}}{\sin^2 \theta} + 2g \right)$$

where

$$x = r(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta).$$

Note that the Laplace-Beltrami operator is

$$\Delta_{\mathbb{S}^2} g = g_{\theta\theta} + \frac{\cos \theta}{\sin \theta} g_\theta + \frac{g_{\phi\phi}}{\sin^2 \theta}.$$

Thus we get

$$(6.1) \quad \Delta_{\mathbb{S}^2} g + 2g = 0.$$

It is known [1] that the eigenvalues of

$$\nabla_{ij}^2 H(n) + \delta_{ij} H(n)$$

are the principal radii of curvature of the surface determined by  $H$ , where  $H$  is the Minkowski support function defined on the sphere and the second order derivatives are taken with respect to an orthonormal frame at  $n \in \mathbb{S}^{N-1}$ . The support function uses the inverse of the Gauss map to parametrize the surface as follows

$$H(n) = G^{-1}(n) \cdot n.$$

Furthermore, the Gauss curvature  $K$  is computed as follows [1]

$$(6.2) \quad \frac{1}{K} = \det(\nabla_{ij}^2 H(n) + \delta_{ij} H(n)).$$

The Gauss map is a local diffeomorphism whenever  $K \neq 0$  [17]. Since  $u_0 = rg$  is harmonic in  $\{u_0 > 0\}$  we infer that  $g$  is smooth on  $\mathbb{S}^2 \cap \{g > 0\}$ .

**6.2. Catenoid is a solution.** In [2] page 110 Alt and Caffarelli constructed a weak solution which is not a minimizer. Their solution can be given explicitly as follows: let

$$x = r(\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$$

and take

$$u(x) = r \max \left( \frac{f(\theta)}{f'(\theta_0)}, 0 \right)$$

where

$$f(\theta) = 2 + \cos \theta \log \left( \frac{1 - \cos \theta}{1 + \cos \theta} \right) = 2 + \cos \theta \log \left( \tan^2 \frac{\theta}{2} \right)$$

and  $\theta_0$  is the unique zero of  $f$  between 0 and  $\frac{\pi}{2}$ . The aim of this section is to show that  $f$  is the support function of catenoid. Recall that the principal radii of curvature of a smooth surface are the eigenvalues of the matrix  $\nabla_{\mathbb{S}^{N-1}}^2 H + \delta_{ij} H$  where the Hessian is taken with respect to the sphere  $\mathbb{S}^{N-1}$  [1]. At each point where the Gauss curvature does not vanish the zero mean curvature condition for  $N = 3$  can be written as

$$\Delta_{\mathbb{S}^2} H + 2H = 0$$

where  $\Delta_{\mathbb{S}^2}$  is the Laplace-Beltrami operator and  $H(n)$  is the value of Minkowski's support function corresponding to the normal  $n \in \mathbb{S}^2$ . From now on let us consider the  $(x, y)$  variables on  $\mathbb{R}^2$ . Recall that by rotating the graph of  $y(x) = a \cosh \frac{x}{a}$  around the  $x$ -axis one obtains a catenoid for some constant  $a$ . Thus it is enough to compute the support function for the graph of  $y$ . Let  $\alpha$  be the angle the tangent line of  $y$  at  $(x, y(x))$  forms with the  $x$ -axis. If  $n$  is the unit normal to the graph of  $y$  then  $n = (-\sin \alpha, \cos \alpha)$  and

$$H(n) = (x, y(x)) \cdot n = -x \sin \alpha + a \cos \alpha \cosh \frac{x}{a}.$$

Noting that the unit tangent at  $(x, y(x))$  is  $(\cos \alpha, \sin \alpha)$  and equating with the slope of tangent line, which is  $(\sinh \frac{x}{a}, 1)$ , we obtain

$$\cos \alpha = \frac{\sinh \frac{x}{a}}{\sqrt{1 + \sinh^2 \frac{x}{a}}}, \quad \sin \alpha = -\frac{1}{\sqrt{1 + \sinh^2 \frac{x}{a}}}.$$

From second equation we get that  $\sinh \frac{x}{a} = \tan \alpha$  and solving the quadratic equation  $e^{2\frac{x}{a}} - 1 = 2e^{\frac{x}{a}} \tan \alpha$  we find that

$$x = a \log \frac{1 + \sin \alpha}{\cos \alpha}, \quad \cosh \frac{x}{a} = \frac{1}{\cos \alpha}.$$

Consequently,

$$H(n) = -\frac{a}{2} \sin \alpha \log \left( \frac{1 + \sin \alpha}{\cos \alpha} \right)^2 + a.$$

Taking  $\alpha = \theta + \frac{\pi}{2}$  we have

$$\frac{1 + \sin \alpha}{\cos \alpha} = \frac{1 + \cos \theta}{-\sin \theta} = \frac{2 \cos^2 \frac{\theta}{2}}{-2 \sin \frac{\theta}{2} \cos \frac{\theta}{2}} = -\cot \frac{\theta}{2}$$

and thus choosing  $a = 2$  the result follows.

**6.3. Minimal immersion.** Consider the parametrization  $\mathcal{X} : U_g \rightarrow \mathbb{R}^3$ , where

$$(6.3) \quad \mathcal{X}(n) = ng(n) + \nabla_{\mathbb{S}^2} g, \quad U_g = \{g > 0\} \subset \mathbb{S}^2.$$

Let  $\mathcal{M}$  be the hypersurface determined by  $\mathcal{X}$ . The spherical part  $g$  of  $u_0$  solves the equation (6.1) and by Theorem 1 [21]  $\mathcal{X}$  determines a smooth map which is either constant or a conformal minimal immersion outside locally finite set of isolated singularities (branch points). Next lemma shows that  $\mathcal{X}$  is an conformal immersion everywhere. Recall that at the singularities we must have

$$(6.4) \quad \mathcal{X}_u \times \mathcal{X}_v = 0, \quad n = (u, v) \in \mathbb{T}U_g$$

and such points are called branch points, see [18] page 314.

**Lemma 6.1.** *Let  $\mathcal{M}$  be the surface determined by  $\mathcal{X}$  in (6.3). Then  $\mathcal{M}$  does not have branch points. In particular,  $\mathcal{X}$  is a conformal minimal immersion.*

**Proof.** Observe that  $\mathcal{X}(n)$  is the gradient of the blow-up  $u_0$  at  $n = \frac{x}{|x|}$ . Indeed,

$$\begin{aligned} (6.5) \quad \mathcal{X}(n) &= \frac{n}{r} rg + \frac{1}{r} \nabla_{\mathbb{S}^2} (rg) = \\ &= \frac{n}{r} u_0(x) + \frac{1}{r} \nabla_{\mathbb{S}^2} u_0 = \\ &= \frac{n}{r} (\nabla u_0(x) \cdot x) + \frac{1}{r} \nabla_{\mathbb{S}^2} u_0 = \\ &= n (\nabla u_0(x) \cdot \frac{x}{|x|}) + \frac{1}{r} \nabla_{\mathbb{S}^2} u_0 = \\ &= \nabla u_0(x). \end{aligned}$$

In particular, the computation above shows that

$$(6.6) \quad \nabla u_0(x) = \nabla u_0\left(\frac{x}{|x|}\right), \quad \nabla_{\mathbb{S}^2} g(n) \perp n,$$

in other words the gradient is homogeneous of degree zero.

Suppose there is  $n \in U_g$  such that  $\mathcal{X}_u(n) \times \mathcal{X}_v(n) = 0$ . Without loss of generality we assume that  $n$  points in the direction of the  $x_1$  axis and we choose the  $(u, v)$  frame such that  $u, v$  point in the  $x_2$  and  $x_3$



directions, respectively. From (6.6) we have that  $\nabla u_0(te_1) = \nabla u_0(e_1), t > 0$  where  $e_1$  is the unit direction of the  $x_1$  axis. Differentiating in  $t$  we obtain  $\partial_{1i}u_0(te_1) = 0, i = 1, 2, 3$  for all  $t > 0$ . Therefore, the Hessian matrix at  $te_1$  has the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \partial_{22}u_0 & \partial_{23}u_0 \\ 0 & \partial_{23}u_0 & \partial_{33}u_0 \end{pmatrix}.$$

By our assumption  $n = e_1$  is a branch point and thus  $\partial_2\mathcal{X} \times \partial_3\mathcal{X} = 0$  at  $e_1$ . There are three possible cases:

**Case 1**  $\mathcal{X}_2 = 0$ : then  $\partial_{22}u_0 = \partial_{23}u_0 = 0$  along the ray  $\ell = \{x \in \{u_0 > 0\} : x = t\varepsilon_1, t > 0\}$ . Moreover,  $\Delta u_0 = 0$  implies that  $\partial_{33}u_0 = 0$ . Hence, the Hessian of  $u_0$  vanishes along  $\ell$ .

**Case 2**  $\mathcal{X}_3 = 0$ : proceeding as above we will arrive at the same conclusion.

**Case 3**  $\mathcal{X}_2$  and  $\mathcal{X}_3$  are linearly dependent: Let  $\sigma \in \mathbb{R}$  such that  $\mathcal{X}_2 = \sigma\mathcal{X}_3$ . This yields  $\partial_{22}u_0 - \sigma\partial_{32}u_0 = 0$  and  $\partial_{23}u_0 - \sigma\partial_{33}u_0 = 0$ . Eliminating  $\partial_{23}u_0 = \partial_{32}u_0$  we get

$$0 = \partial_{22}u_0 - \sigma\partial_{32}u_0 = \partial_{22}u_0 - \sigma^2\partial_{33}u_0 = (1 + \sigma^2)\partial_{22}u_0.$$

Consequently,  $\partial_{22}u_0 = 0$  along  $\ell$  and from  $\Delta u_0 = 0$  we again conclude that the Hessian vanishes along  $\ell$ .

Since  $\ell$  has positive  $\mathcal{H}^1$  measure then by Theorem 2.1 [12] we must have  $u_0 \equiv 0$  which is a contradiction. Thus the proof is complete.  $\square$

The absence of branch points does not rule out the possibility of self-intersection. Therefore we need to prove that  $\mathcal{M}$  is embedded.

**6.4. Dual cones and center of mass.** Recall that by Theorem 5.9 and Lemmata 5.8, 5.10 we know that under the assumptions of Theorem C  $\partial\{u_0 > 0\} \setminus \{0\}$  is a union of smooth convex cones  $\mathcal{C}_1$  and  $\mathcal{C}_2$ . We define the dual cones as follows

$$\mathcal{C}_i^* = \partial\{y \in \mathbb{R}^3 : x \cdot y \leq 0, x \in \mathcal{C}_i\}, \quad i = 1, 2.$$

It is well-known that the dual of a convex cone is also convex, [23] page 35. Let us put  $\gamma_i = \mathbb{S}^2 \cap \mathcal{C}_i^*$ .

**Lemma 6.2.** *The largest principal curvature of  $\mathcal{C}_i \setminus \{0\}$  is strictly positive.*

**Proof.** To fix the ideas we prove the statement for  $\mathcal{C}_1$ . Note that one of the principal curvatures of  $\mathcal{C}_1 \setminus \{0\}$  is zero because  $\mathcal{C}_1$  is a cone and  $\mathcal{C}_1 \setminus \{0\}$  is smooth, see Theorem B. Let  $\kappa(p)$  be the largest principal curvature at  $p \in \mathcal{C}_1 \setminus \{0\}$ . Suppose there is  $p$  such that  $\kappa(p) = 0$ . Since  $u_0 = 0$  on a piece of regular free boundary near  $p$  then we can extend  $u_0$  to a solution  $\tilde{u}_0$  of uniformly elliptic equation  $\partial_i(a_{ij}(x)\partial_j u_0)$  in some neighborhood of  $p$  where  $a_{ij}$  is a uniformly elliptic matrix with  $C^\infty$  entries (in fact,  $a_{ij}(x) = \delta_{ij}$  if  $x \in \{u_0 > 0\}$ ). Choose the coordinate system at  $p$  so that  $x_1$  points in the outward normal direction at  $p$  (into  $\{u_0 \equiv 0\}$ ),  $x_2$  is tangential at  $p$  and is the principal direction corresponding to  $\kappa(p)$ . Then we have that  $\nabla u_0(p) = e_1$ , the unit direction of  $x_1$  axis and the mean curvature of  $\mathcal{C}_1$  at  $p$  vanishes because we assumed that  $\kappa(p) = 0$ . Writing the mean curvature at  $p$  in terms of the derivatives of  $u_0$  we have

$$0 = \frac{\nabla u_0 D^2 u_0 (\nabla u_0)^T - |\nabla u_0|^2 \Delta u_0}{2|\nabla u_0|^3} = \partial_{11}u_0$$

implying that  $\partial_{11}u_0 = 0$ . Moreover, since  $u_0$  is homogeneous of degree one then  $\nabla u_0 = e_1$  along the  $x_1$  axis. This yields  $\partial_{13}u_0 = \partial_{23}u_0 = \partial_{33}u_0 = 0$  along the  $x_1$  axis. From the harmonicity of  $u_0$  (note that  $a_{ij}(p) = \delta_{ij}$ ) it follows that  $\partial_{22}u_0 = 0$  along the  $x_1$  axis. Summarizing, we have that along the points of the  $x_1$  axis the Hessian of  $u_0$  has the following form

$$\begin{pmatrix} 0 & \partial_{12}\tilde{u}_0 & 0 \\ \partial_{12}\tilde{u}_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Finally, letting  $\sigma(t), t \in (-\delta, \delta)$  be the parametrization of the curve along which the  $x_1x_2$  plane intersects with  $\mathcal{C}_1$  and differentiating  $|\nabla u_0(\sigma(t))| = 1$  in  $t$  we get that at  $p$  one must have

$$0 = e_1 \begin{pmatrix} 0 & \partial_{12}\tilde{u}_0 & 0 \\ \partial_{12}\tilde{u}_0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} e_2 = \partial_{12}u_0(p).$$

Thus, the Hessian  $D^2u_0$  vanishes along the  $x_1$  axis. Applying [12] Theorem 2.1 and Lemma 2.2 to the function  $w = u_0 - x_1$ , we get that  $w$  is identically zero, or equivalently  $u_0 = x_1$  which is a contradiction because under our assumption  $\partial\{u_0 > 0\}$  consists of  $\mathcal{C}_1$  and  $\mathcal{C}_2$ .  $\square$

**Remark 6.3.** It follows from Lemma 6.2 and Theorem B that there are two positive constants  $\kappa_0, \kappa_1$  such that

$$0 < \kappa_0 \leq \kappa(p) \leq \kappa_1, \quad p \in \partial\{u_0 > 0\} \setminus \{0\}$$

here  $\kappa(p)$  is the largest curvature at the free boundary points away from the origin.

**Lemma 6.4.** Let  $\mathcal{C}_1^*, \mathcal{C}_2^*$  be the dual cones as above. Then we have

- (i)  $\partial\mathcal{M}$  is differentiable and there are two positive constants  $\kappa_0^*, \kappa_1^*$  such that the largest curvature  $\kappa^*(p)$  of  $\mathcal{C}_i^* \setminus \{0\}$  satisfies  $\kappa_0^* \leq \kappa^*(p) \leq \kappa_1^*$ ,
- (ii) there is  $\delta > 0$  small such that  $\partial B_{1-\delta} \cap \mathcal{M}$  defines a convex cone,
- (iii)  $\mathcal{M}$  is star-shaped with respect to the origin and hence embedded.

**Proof.** Suppose that  $\mathcal{C}_1^*$  is not differentiable at some  $z \neq 0$ . Then  $\mathcal{C}_i$  must have a flat piece. Indeed, if  $n_1, n_2$  are the normals of two supporting hyperplanes of  $\mathcal{C}_i^*$  at  $z$  then the unit vectors  $n_t = \frac{tn_1 + (1-t)n_2}{|tn_1 + (1-t)n_2|}$ , define a support function at  $z$  for every  $t \in (0, 1)$ . Since the vectors  $n_t$  lie on the same plane then  $\mathcal{C}_1$  must have a flat piece. The unique continuation theorem implies that the free boundary is a hyperplane and cannot have singularities. Now the desired estimate follows from Remark 6.3 and the definition of dual cone. The first claim is proved.

Let  $k_1, k_2$  be the principal curvatures of  $\mathcal{M}$ , then  $k_1 + k_2 = 0$  and the Gauss curvature is  $K = -k_1^2 = -k_2^2$ . Since  $\mathcal{M}$  is an immersion then from (6.2) and the smoothness of  $g = \nabla u_0$  in  $U_g$  we see that  $K \neq 0$ . Furthermore, there is a tame constant  $c_0 > 0$  such that  $k_i^2 \geq c_0, i = 1, 2$  at every point of  $\mathcal{M}$ . Thus  $\mathcal{M}$  is fibred by  $\partial B_{1-\delta}$  for  $\delta > 0$  small. We claim that  $|\mathcal{X}(n)| > 0, n \in \overline{U_g}$ . Clearly this is true if  $n \in \partial U_g$  where  $|\mathcal{X}(n)| = 1$ . Suppose there is  $n \in U_g$  such that  $\mathcal{X}(n) = 0$ . Since  $\mathcal{X}(n) = ng + \nabla_{\mathbb{S}^2}g$  it follows that  $g(n) = 0$

but this is impossible since  $n \in \{g > 0\} = U_g$ . From  $g(n) = \mathcal{X}(n) \cdot n > 0, n \in U_g$  it follows that  $\mathcal{M}$  is starshaped with respect to the origin and hence embedded.  $\square$

Let  $n \in U_g$  then from  $\mathcal{X}(n) = \nabla u_0(n)$  it follows that

$$|\mathcal{X}(n)|_{\partial\{u_0>0\}} = |\nabla g| = \sqrt{2M}.$$

Since by previous Lemma 6.4  $\mathcal{M}$  is differentiable along  $\gamma_i$  we see that the contact angle  $\alpha$  between  $\mathcal{M}$  and  $\mathbb{S}^2$  is

$$\cos \alpha = n \cdot \frac{\mathcal{X}(n)|_{\partial\{u_0>0\}}}{\sqrt{2M}} = g(n)|_{\partial\{u_0>0\}} = 0.$$

Thus, the minimal surface defined by  $g$  is inside of the sphere of radius  $\sqrt{2M}$  in view of Lemma 7.7  $|\nabla u_0|^2 = g^2 + |\nabla g|^2 \leq 2M$ . Moreover,  $\mathcal{M}$  is tangential to  $\mathcal{C}_1^*$  and  $\mathcal{C}_2^*$  along  $\mathbb{S}^2$  since  $n \perp \nabla_{\mathbb{S}^2} g$  by (6.6).

We recall the definition from [19] page 47.

**Definition 6.1.** *We say that  $\mathcal{M}$  is of topological type  $[\epsilon, r, \chi]$  if it has orientation  $\epsilon$ , Euler characteristic  $\chi$ , and  $r$  boundary curves. Here  $\epsilon = \pm 1$ , where  $+1$  means that  $\mathcal{M}$  is orientable and  $\epsilon = -1$  non-orientable. For orientable surface the Euler characteristic is defined by the relation  $\chi = 2 - 2g - r$  where  $g$  is the genus of  $\mathcal{M}$ .*

Now the first part of Theorem C follows from Nitsche's theorem, see page 2 [19]. Moreover, the only stationary surfaces of disk type are the totally geodesic disks and the spherical cups. From Lemma 5.1 it follows that if  $u_0 = rg$  and  $\text{supp } g$  is a disk then  $u_0$  is a half plane.

In view of Lemma 6.4 (iii) the proof of Theorem C can be deduced from the result of Nitsche [20] but we will sketch a shorter proof based on Aleksandrov's moving plane method and Serrin's boundary lemma. We reformulate Theorem C as follows

**Lemma 6.5.** *Let  $\mathcal{M}$  be of topological type  $[1, 2, 0]$ , i.e. a ring-type minimal surface. Then  $\mathcal{M}$  is a catenoid.*

**Proof.** By Lemmata 6.1 and 6.4 (iii)  $\mathcal{X}$  is a conformal minimal immersion (in fact,  $\mathcal{M}$  is embedded). Let  $\partial\mathcal{M} = \gamma_1 \cup \gamma_2$ . Then applying Stokes formula we have

$$(6.7) \quad \int_{\mathcal{M}} \Delta_{\mathcal{M}} \mathcal{X} = \int_{\partial\mathcal{M}} n^* ds = \int_{\gamma_1} n^* + \int_{\gamma_2} n^* ds$$

where  $n^*$  is the outward conormal, i.e.  $n^*$  tangent to  $\mathcal{M}$  and normal to  $\partial\mathcal{M}$ , see [11] page 81. Since  $\mathcal{X}$  is minimal then  $\Delta_{\mathcal{M}} \mathcal{X} = 0$ . Thus

$$(6.8) \quad \int_{\gamma_1} n^* ds + \int_{\gamma_2} n^* ds = 0.$$

Since  $\mathcal{M}$  is tangential to  $\mathcal{C}_i^*$  it follows that the conormal  $n^*$  on  $\gamma_i$  points in the direction of the generatrix of the dual cone  $\mathcal{C}_i^*$ . Observe that the sums  $S_m = \sum_{k=0}^m n_k^{*i}(s_{k+1} - s_k), n_k^{*i} \in \mathcal{C}_i^*$  approximate the boundary integrals, consequently the vector  $S_m$  is strictly inside of the cone  $\mathcal{C}_i^*$  and in the limit converges to the centre of mass of  $\gamma_i$  computed with respect of the origin (the vertex of the cone). In view of (6.8) there is a diameter of  $\mathbb{S}^2$  strictly inside of both dual cones  $\mathcal{C}_1^*$  and  $\mathcal{C}_2^*$ .

Without loss of generality we assume that the diameter passes through the north and south poles. Now we can apply Aleksandrov's moving plane method and Serrin's boundary point lemma to finish the proof. Let  $\Pi_t$  be a family of planes containing  $x_1$  axis and  $t$  measures the angle between  $\Pi_t$  and  $x_3$  axis.

Now start rotate  $\Pi_t$  about  $x_1$  axis starting from a position when  $\Pi_t$  is a support hyperplane to one of the cones  $\mathcal{C}_1^*, \mathcal{C}_2^*$  and  $\Pi_t \cap \mathcal{C}_i^* \neq \emptyset, i = 1, 2$ .

**Case 1:** If the first touch of  $\mathcal{M}$  and its reflection  $\tilde{\mathcal{M}}$  with respect to the plane  $\Pi_t$  happens at an interior point of  $\mathcal{M}$  then from the maximum principle it follows that  $\mathcal{M} = \tilde{\mathcal{M}}$ .

By Lemma 6.4, both dual cones are strictly convex. Moreover, we claim that for  $\delta$  small the cones generated by  $\mathcal{M} \cap \partial B_{1-\delta}$  are convex, otherwise the inflection point would propagate to  $\mathcal{C}_i^*$ .

The two remaining possibilities are:

**Case 2:** if  $\mathcal{M}$  and its reflection  $\tilde{\mathcal{M}}$  first touch at a boundary point where  $\partial\mathcal{M}$  is perpendicular to  $\Pi_t$

**Case 3:** if  $\mathcal{M}$  and its reflection  $\tilde{\mathcal{M}}$  first touch at a boundary point where  $\partial\mathcal{M}$  not lying on  $\Pi_t$ .

We cannot directly apply Serrin's boundary point lemma [25] because  $\partial\mathcal{M}$  is only  $C^{1,1}$  by Lemma 6.4. However, from the fibering of  $\mathcal{M}$  near  $\partial\mathcal{M}$  we conclude that  $\tilde{g} \leq g$  near the contact point, where  $\tilde{g}$  is the support function of  $\tilde{\mathcal{M}}$ . Thus  $\tilde{u} = r\tilde{g} \leq rg = u$ . Hence applying Serrin's boundary point lemmas to the harmonic functions  $\tilde{u}$  and  $u$  we conclude that  $\mathcal{M} = \tilde{\mathcal{M}}$ .

Choosing  $\Pi_t$  to be an arbitrary family passing through a line perpendicular to the diameter it follows that  $\gamma_1, \gamma_2$  are circles and (6.8) forces them to lie on parallel planes. Applying Corollary 2 [24] we infer that  $\mathcal{M}$  is a part of catenoid.  $\square$

## 7. APPENDIX

This section contains some well known results on the singular perturbation problem. We begin with the uniform Lipschitz estimates of Luis Caffarelli, see [6] for the proof.

**Proposition 7.1.** *Let  $\{u_\varepsilon\}$  be a family of solution of  $(\mathcal{P}_\varepsilon)$  then there is a constant  $C$  depending only on  $N, \|\beta\|_\infty$  and independent of  $\varepsilon$  such that*

$$(7.1) \quad \|\nabla u_\varepsilon\|_{L^\infty(B_{\frac{1}{2}})} \leq C.$$

As a consequence we get that one can extract converging sequences  $\{u_{\varepsilon_n}\}$  of solutions of  $(\mathcal{P}_\varepsilon)$  such that they converges to stationary points of the Alt-Caffarelli problem.

**Proposition 7.2.** *Let  $u_\varepsilon$  be a family of solutions to  $(\mathcal{P}_\varepsilon)$  in a domain  $\mathcal{D} \subset \mathbb{R}^N$ . Let us assume that  $\|u_\varepsilon\|_{L^\infty(\mathcal{D})} \leq A$  for some constant  $A > 0$  independent of  $\varepsilon$ . For every  $\varepsilon_n \rightarrow 0$  there exists a subsequence  $\varepsilon_{n'} \rightarrow 0$  and  $u \in C_{loc}^{0,1}(\mathcal{D})$ , such that*

- (i)  $u_{\varepsilon_{n'}} \rightarrow u$  uniformly on compact subsets of  $\mathcal{D}$ ,
- (ii)  $\nabla u_{\varepsilon_{n'}} \rightarrow \nabla u$  in  $L_{loc}^2(\mathcal{D})$ ,
- (iii)  $u$  is harmonic in  $\mathcal{D} \setminus \partial\{u > 0\}$ .

**Proof.** See Lemma 3.1 [10].  $\square$

Next, we recall the estimates for the slopes of some global solutions.

**Proposition 7.3.** *Let  $u$  be as in Proposition 7.2. Then the following statements hold true:*

- (i)  $u$  is Lipschitz,
- (ii) if  $u_{\varepsilon_j} \rightarrow u = \alpha x_1^+ + \bar{\alpha} x_1^-$  locally uniformly, then  $0 \leq \alpha \leq \sqrt{2M}$ , see Proposition 5.2 [10],
- (iii) if  $u_{\varepsilon_j} \rightarrow u = \alpha x_1^+ - \gamma x_1^- + o(|x|)$  and  $\gamma > 0$  then  $\alpha^2 - \gamma^2 = \sqrt{2M}$ , see Proposition 5.1 [10]. In this lemma the essential assumption is that  $\gamma > 0$ .

**Remark 7.4.** *Observe that if  $u(x) = \alpha x_1^+ + \bar{\alpha} x_1^-$  then we must necessarily have that  $\alpha = \bar{\alpha} \leq \sqrt{2M}$ , see Proposition 5.3 [10]. In this case the interior of the zero set of  $u$  is empty. Thus one might have wedge-like solution.*

Using Proposition 7.1 we can extract a sequence  $u_{\varepsilon_j}$  for some sequence  $\varepsilon_j$  such that  $u_{\varepsilon_j} \rightarrow u$  uniformly in  $B_{\frac{1}{2}}$ , see Proposition 7.2. Let  $u$  be a limit and  $0 < \rho_j \downarrow 0$  and  $u_j(x) = \frac{u(x_0 + \rho_j x)}{\rho_j}$ ,  $x_0 \in \partial\{u > 0\}$ . Thanks to Proposition 7.3(i) we can extract a subsequence, still labeled  $\rho_j$ , such that  $u_j$  converges to some function  $u_0$  defined in  $\mathbb{R}^N$ . The function  $u_0$  is called a blow-up limit of  $u$  at the free boundary point  $x_0$  and it depends on  $\{\rho_j\}$ .

The two propositions to follow establish an important property of the blow-up limits, namely that the first and second blow-ups of  $u$  can be obtained from  $(\mathcal{P}_\varepsilon)$  for a suitable choice of parameter  $\varepsilon$ . Observe that the scaled function  $\nabla(u_{\varepsilon_j})_{\lambda_n}$  verifies the equation

$$\Delta(u_{\varepsilon_j})_{\lambda_j} = \frac{\lambda_j}{\varepsilon_j} \beta \left( \frac{\lambda_j}{\varepsilon_j} (u_{\varepsilon_j})_{\lambda_j} \right).$$

Taking  $\delta_j = \frac{\varepsilon_j}{\lambda_j} \rightarrow 0$  we see that  $(u_{\varepsilon_j})_{\lambda_j}$  is solution to  $\Delta u_{\delta_j} = \beta_{\delta_j}(u_{\delta_j})$ .

**Proposition 7.5.** *Let  $u_{\varepsilon_j}$  be a family of solutions to  $(\mathcal{P}_\varepsilon)$  in a domain  $\mathcal{D} \subset \mathbb{R}^N$  such that  $u_{\varepsilon_j} \rightarrow u$  uniformly on  $\mathcal{D}$  and  $\varepsilon_j \rightarrow 0$ . Let  $x_0 \in \mathcal{D} \cap \partial\{u > 0\}$  and let  $x_n \in \partial\{u > 0\}$  be such that  $x_n \rightarrow x_0$  as  $n \rightarrow \infty$ . Let  $\lambda_n \rightarrow 0$ ,  $u_{\lambda_n}(x) = (1/\lambda_n)u(x_n + \lambda_n x)$  and  $(u_{\varepsilon_j})_{\lambda_n} = (1/\lambda_n)u_{\varepsilon_j}(x_n + \lambda_n x)$ . Assume that  $u_{\lambda_n} \rightarrow U$  as  $n \rightarrow \infty$  uniformly on compact subsets of  $\mathbb{R}^N$ . Then there exists  $j(n) \rightarrow \infty$  such that for every  $j_n \geq j(n)$  there holds that  $\varepsilon_j/\lambda_n \rightarrow 0$  and*

- $(u_{\varepsilon_{j_n}})_{\lambda_n} \rightarrow U$  uniformly on compact subsets of  $\mathbb{R}^N$ ,
- $\nabla(u_{\varepsilon_{j_n}})_{\lambda_n} \rightarrow \nabla U$  in  $L_{loc}^2(\mathbb{R}^N)$ ,
- $\nabla u_{\lambda_n} \rightarrow \nabla U$  in  $L_{loc}^2(\mathbb{R}^N)$ .

**Proof.** See Lemma 3.2 [10]. □

Finally, recall that the result of previous proposition extends to the second blow-up.

**Proposition 7.6.** *Let  $u_{\varepsilon_j}$  be a solution to  $(\mathcal{P}_\varepsilon)$  in a domain  $\mathcal{D}_j \subset \mathcal{D}_{j+1}$  and  $\cup_j \mathcal{D}_j = \mathbb{R}^N$  such that  $u_{\varepsilon_j} \rightarrow U$  uniformly on compact sets of  $\mathbb{R}^N$  and  $\varepsilon_j \rightarrow 0$ . Let us assume that for some choice of positive numbers  $d_n$  and points  $x_n \in \partial\{U > 0\}$ , the sequence*

$$U_{d_n}(x) = \frac{1}{d_n} U(x_n + d_n x)$$

converges uniformly on compact sets of  $\mathbb{R}^N$  to a function  $U_0$ . Let

$$(u_{\varepsilon_j})_{d_n} = \frac{1}{d_n} u_{\varepsilon_j}(x_n + d_n x).$$

Then there exists  $j(n) \rightarrow \infty$  such that for every  $j_n \geq j(n)$ , there holds  $\varepsilon_{j_n}/d_n \rightarrow 0$  and

- $(u_{\varepsilon_{j_n}})_{d_n} \rightarrow U_0$  uniformly on compact subsets of  $\mathbb{R}^N$ ,
- $\nabla(u_{\varepsilon_j})_{d_n} \rightarrow \nabla U_0$  in  $L^2_{loc}(\mathbb{R}^N)$ .

**Proof.** See Lemma 3.3 [10]. □

**Lemma 7.7.** *Let  $u \geq 0$  be as in Proposition 7.2. Then*

$$\limsup_{x \rightarrow x_0, u(x) > 0} |\nabla u(x)| \leq \sqrt{2M}.$$

**Proof.** To fix the ideas we let  $x_0 = 0$  and  $l = \limsup_{x \rightarrow 0, u(x) > 0} |\nabla u(x)|$ . Suppose  $l > 0$ , otherwise we are done. Choose a sequence  $z_k \rightarrow 0$  such that  $u(z_k) > 0$  and  $|\nabla u(z_k)| \rightarrow l$ . Setting  $\rho_k = |y_k - z_k|$ , where  $y_k \in \partial\{u > 0\}$  is the nearest point to  $z_k$  on the free boundary and proceeding as in the proof of [4] Lemma 3.4 we can conclude that the blow-up sequence  $u_k(x) = \rho_k^{-1} u(z_k + \rho_k x)$  has a limit  $u_0$  (at least for a subsequence, thanks to Proposition 7.1) such that  $u_0(x) = l x_1, x_1 > 0$  in a suitable coordinate system. Moreover, by Proposition 7.5 it follows that  $u_0$  is a limit of some  $u_{\lambda_j}$  solving  $\Delta u_{\lambda_j} = \beta_{\lambda_j}(u_{\lambda_j})$  in  $B_{r_j}, r_j \rightarrow \infty$ . If there is a point  $z \in \{x_1 = 0\}$  and  $r > 0$  such that  $u_0 > 0$  in  $B_r(z) \cap \{x_1 < 0\}$  then near  $z$  we must have  $u_0(x) = l(x - z)_1^+ + l(x - z)_1^- + o(x - z)$ , see Remark 7.4. Applying the unique continuation theorem to  $u_0(x) - u_0(-x_1, x_2, \dots, x_n)$  we see that  $u_0 = l(-x_1)^+, x_1 < 0$ . Thus recalling Remark 7.4 again we infer that  $l \leq \sqrt{2M}$ . □

Finally, we mention a useful identity for the solutions  $u_\varepsilon$ , see equation (5.2) [10]: Let  $u_\varepsilon$  be a solution of  $(\mathcal{P}_\varepsilon)$  then for any  $\phi \in C_0^\infty(B_1)$  there holds

$$(7.2) \quad \int \left( \frac{|\nabla u_\varepsilon|^2}{2} + B(u_\varepsilon/\varepsilon) \right) \partial_1 \phi = \int \sum_k \partial_k u_\varepsilon \partial_1 u_\varepsilon \partial_k \phi.$$

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